Question. *Consider the function*

$$f(x,y) = \begin{cases} \dfrac{x^3y - xy^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

(a) *Find* $f_x(x,y)$ *and* $f_y(x,y)$ *when* $(x,y) \neq (0,0)$.
(b) *Find* $f_x(0,0)$ *and* $f_y(0,0)$. *You will need to use the definitions of partial derivatives at a point to do this.*
(c) *Find* $f_{xy}(0,0)$ *and* $f_{yx}(0,0)$. *You will again need to use the definition of partial derivatives at a point.*
(d) *Reconcile this with Clairaut's theorem.*

Answer.

(a) *Away from* $(0,0)$, *there are no problems, so we just compute the partial derivatives as usual*

$$\begin{aligned} f_x(x,y) &= \frac{(3x^2y - y^3)(x^2+y^2) - (2x)(x^3y - xy^3)}{(x^2+y^2)^2} \\ &= \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2} \end{aligned}$$

and, by symmetry (switching x *and* y *in the function* f *only creates a minus sign, so we can compute* f_y *by switching the variables in* f_x *and adding a minus sign)*

$$f_y(x,y) = \frac{x^5 - 4x^3y^2 - xy^4}{(x^2+y^2)^2}$$

(b) *To compute the derivatives at zero, because of the way the function is defined, we must use the limit definitions:*

$$\begin{aligned} f_x(0,0) &= \lim_{h \longrightarrow 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h \longrightarrow 0} \frac{\frac{h^3 0 - h0^3}{h^2+0^2} - 0}{h} \\ &= \lim_{h \longrightarrow 0} \frac{0}{h} = 0 \\ f_y(0,0) &= \lim_{h \longrightarrow 0} \frac{f(0,0+h) - f(0,0)}{h} = \lim_{h \longrightarrow 0} \frac{\frac{0^3 h - 0h^3}{0^2+h^2} - 0}{h} \\ &= \lim_{h \longrightarrow 0} \frac{0}{h} = 0 \end{aligned}$$

(c) *Once again, we use the limit definitions to compute these derivatives*

$$\begin{aligned} f_{xy}(0,0) &= \lim_{h \longrightarrow 0} \frac{f_x(0,0+h) - f_x(0,0)}{h} = \lim_{h \longrightarrow 0} \frac{\frac{0^4h + 40^2h^3 - h^5}{(0^2+h^2)^2} - 0}{h} \\ &= \lim_{h \longrightarrow 0} \frac{-h}{h} = -1 \\ f_{yx}(0,0) &= \lim_{h \longrightarrow 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h \longrightarrow 0} \frac{\frac{h^5 - 4h^2 0^2 - h0^4}{h^2+0^2} - 0}{h} \\ &= \lim_{h \longrightarrow 0} \frac{h}{h} = 1 \end{aligned}$$

(d) *It might seem at first that since* $f_{xy}(0,0) = -1$ *and* $f_{yx}(0,0) = 1$, *we have found a contradiction to Clairaut's theorem, but remember that the condition in Clairaut's theorem is that all of the second partial derivatives must be continuous (that is,* f *is a* C^2 *function). So, the question now becomes, "is* f_{xy} *continuous?" (we could equally well use* f_{yx}*). First, we find* $f_{xy}(x,y)$ *away from* $(0,0)$. *This just involves taking the derivative as usual of* f_x *with respect to* y. *We find*

$$f_{xy}(x,y) = \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2+y^2)^3}$$

Taking the limit to the origin along $y = mx$ *yields*

$$\begin{aligned}
\lim_{(x,mx)\longrightarrow(0,0)} f_{xy}(x,y) &= \lim_{x\longrightarrow 0} \frac{x^6 + 9m^2x^6 - 9m^4x^6 - m^6x^6}{(x^2+m^2x^2)^3} \\
&= \lim_{x\longrightarrow 0} \frac{x^6(1 + 9m^2 - 9m^4 - m^6)}{x^6(1+m^2)^3} \\
&= \frac{1 + 9m^2 - 9m^4 - m^6}{(1+m^2)^3}
\end{aligned}$$

which depends on m, *therefore the limit does not exist. This means* f_{xy} *is not continuous at* $(0,0)$, *and in particular, Clairaut's theorem fails at the origin.*